Differentials (review and examples)

If $f: X \to Y$ is a morphism of schemes, $d: \mathcal{O}_X \to \Omega_{X/Y}$ the universal element of $Der_{X/Y}(\Omega_{X/Y})$. This characterizes the pair $(\Omega_{X/Y}, d)$ up to unique isomorphism. We constructed this pair explicitly by taking

$$\Omega_{X/Y} = I_{X/Y}/I^2_{X/Y},$$

where $I_{X/Y}$ is the ideal of the diagonal $X \to X \times_Y X$, and where da is the class of $p_2^*(a) - p_1^*(a)$.

Theorem 1 *Let $T \to T'$ be a first order thickening of Y-schemes with ideal I, and let $h: T \to X$ be a Y-morphism. Then under the evident action, $Def_h(T')$ becomes a pseudo-torsor under $Der_{X/Y}(h_* I)$.*

There are two useful exact sequences that we need to recall.

Theorem 2 *Let $g: X \to Z$ and $h: Z \to Y$ be morphisms, with $f = h \circ g$. Then there is a natural exact sequence*

$$g^*\Omega_{Z/Y} \to \Omega_{X/Y} \to \Omega_{X/Z} \to 0$$

Theorem 3 *If $i: Z \to X$ is a closed immersion, there is an exact sequence*

$$I/I^2 \to i^*\Omega_{X/Y} \to \Omega_{Z/Y} \to 0$$

where I is the ideal of Z and the first map is induced from the derivation $d: I \to \Omega^1_{X/Y}$.

Example 4 *Suppose that $Z = Y$ and the composite $Z \to X \to Y = \mathrm{id}_Y$. Then the sequence above induces an isomorphism*

$$I/I^2 \to i^*\Omega_{X/Y}.$$

Indeed, it is clear that $\Omega_{Y/Y} = 0$, and it remains only to prove that the map

$$\overline{\mathrm{d}}: I/I^2 \to i^*\Omega_{X/Y}$$

is injective. For this it suffices to show that there is a map $h: \Omega_{X/Y} \to I/I^2$ with $h \circ \overline{d} = \mathrm{id}$. By the universal mapping property, it suffices to construct a $D \in Der_{X/Y}(I/I^2)$ such that $D(a)$ is the image $\overline{a}$ of a in I/I^2 for each $a \in I$. In fact, for any $a \in \mathcal{O}_X$, $a - f^*i^*(a) \in I$, and we let $D(a)$ be its image in I/I^2. Then D is the desired derivation. (This can be checked directly, or by using a geometric argument involving the first infinitesimal neighborhood of i.

Example 5 *If Y is the spectrum of a field k and X is a k-scheme of finite type and $Z := \{x\}$, where x is a closed point of X, then $k(x)$ is a finite extension of k. If $k = k(x)$, the above result gives us an isomorphism*

$$m_x/m_x^2 \cong \Omega_{X/k}(x).$$

Thus in this case the Zariski tangent space of X at x, the k-dual of m_x/m_x^2, or equivalently the set of deformations of the of the inclusion $x \to X$ to the dual numbers $D_k(\epsilon)$, becomes identified with the set of maps $\Omega_{X/k} \to i_*k(x)$, that is, with the fiber of $\mathbf{V}(\Omega_{X/k})$ over x. In general, $\mathbf{V}\Omega_{X/Y}$ is called the tangent space (or bundle) of X/Y.

Corollary 6 *Let X/k be a scheme locally of finite type, where k is algebraically closed. Then the dimension of m_x/m_x^2 is an uppersemicontinuous function on the set of closed points of X.*

Proof: This is because $\Omega_{X/k}$ is a quasi-coherent sheaf of finite type on X (hence coherent, since X is noetherian), and it follows from Nakayama's lemma that the dimension of $\Omega_{X/k}(x)$ is upper semicontinous. In fact: $\Box$

Lemma 7 *Let X be a scheme and let E be a quasi-coherent sheaf of $\mathcal{O}_X$-modules which is locally finitely generated. Then the dimension of $E(x)$ is uppersemicontinuous on X. If E is locally free, it is in fact locally constant, and the converse holds if X is reduced.*

Proof: Let x be a point of X, and let $(e_1(x), \dots e_d(x))$ be a basis for $E(x) := E_x/m_{X,x}E_x$. There exist an open affine neighborhood U of x and sections $(e_1, \dots, e_d)$ of $E(U)$, such that the image of e_i in $E(x)$ is $e_i(x)$. Replace X by U and let $(\mathcal{O}_X)^n \to E$ be the corresponding map. It follows from Nakayama's lemma that the induced map on $(\mathcal{O}_{X,x})^n \to E_x$ is surjective, and hence that is is surjective in some neighborhood of x. Then $dimE(x') \leq dimE(x)$ for all x' in this neighborhood. Suppose the dimension is in fact constant. We may assume it is constant and that X is affine, say $X = \operatorname{Spec} A$, and E corresponds to a finitely generated A-module M. We have constructed a surjective map $A^n \to M$, where n is the dimension of $M \otimes k(x)$ for every $x \in \operatorname{Spec} A$. It follows that the map $k(x)^n \to M \otimes k(x)$ is bijective for every x. Let $K \subseteq A^n$ be the kernel of $A^n \to M$, and observe that any coordinate of any element of the kernel maps to zero in A_p/PA_P for every prime P. Since A is reduced, the intersection of all the primes is zero, so $K = 0$. $\Box$

Example 8 The map $I/I^2 \to i^*\Omega_{X/Y}$ might not be injective, even if I is the maximal ideal corresponding to a closed point of a scheme X of finite type over $Y = \operatorname{Spec} k$. For example, let k be a field of characteristic p which is not perfect, with an element t which is not a pth power, let $X := \operatorname{Spec} k[X]$ and let I be the ideal generated by $f := X^p - a$. Since this polynomial is irreducible, I is maximal and corresponds to a closed point x. But $df = 0$, so the map in this case is zero.